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Generations of Quarks and Leptons
from Noncompact Horizontal Symmetry

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Abstract

The three chiral generations of quarks and leptons may be generated through a spontaneous breakdown of the noncompact horizontal gauge symmetry G_H which governs, together with the standard gauge symmetry $SU(3) \times SU(2) \times U(1)$, the world in a vectorlike manner. In a framework of supersymmetric theory, the simplest choice $G_H = SU(1,1)$ works quite well for this scenario in which quarks, leptons and Higgses belong to infinite dimensional unitary representation of $SU(1,1)$. The relevance of the scenario to the hierarchical structure of their Yukawa coupling matrices are discussed.

§1. Introduction

One of the remarkable, or even puzzling, facts in the low energy particle physics is the well-regulated repetition of three generations of quarks and leptons

$$q_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}, \quad \bar{u}_i, \quad \bar{d}_i, \quad \ell_i \equiv \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}, \bar{e}_i, \qquad i = 1, 2, 3. \tag{1}$$

Also mysterious is the well-ordered hierarchical structure of the coupling matrices (y_u, y_d, y_e) of their Yukawa interactions with Higgs scalars h and h'

$$y_u^{ij} \, \bar{u}_i \, q_j \, h + y_d^{ij} \, \bar{d}_i \, q_j \, h' + y_e^{ij} \, \bar{e}_i \, \ell_j \, h'. \tag{2}$$

Many attempts have been made to find out the basic structure of Nature lying behind such characteristics of the low energy world.[1,2] Especially the systematic analyses of the Yukawa coupling matrices have been extensively made.[3] Nevertheless we are still far from the satisfactory understanding of what is realized in Nature.

Although supersymmetry is now seriously expected to play an important role in the physics beyond the standard model,[4] we are not yet aware of its relevance to these problems. What is worse, in the supersymmetric theory , we must further understand the reason why the baryon-number and lepton-number violating Yukawa-type interactions,

$$\ell\ell\bar{e} + q\ell\bar{d} + \bar{d}\bar{d}\bar{u} + h'h'\bar{e} , \tag{3}$$

are strongly suppressed in the superpotential.[5] Even in the superstring theory[6] at hand, these are only the criteria for the selection of adequate vacuum from tremendous number of string vacua. It does not give any profound understanding of what is realized in Nature.

One definite approach to these problems, which inquires an inter-generation structure of quarks and leptons, is to invoke a symmetry which governs the generation, that is, horizontal symmetry.[2] Some of the above problems may be understood as a direct consequence of the symmetry, and the others may be attributed to the spontaneous breakdown of the symmetry.

It has been for a long time the belief in particle physics that Nature is as symmetric as possible at the fundamental level. If the horizontal symmetry has any essential responsibility to generations, it cannot be irresponsible to the left-right asymmetry of the low energy physics, which is nothing but the existence of chiral generation itself. It will then be a reasonable expectation that Nature is left-right symmetric (vectorlike)[7] at the fundamental level, and any asymmetry, even the number of generations, is due to the spontaneous breakdown of the horizontal symmetry. The horizontal symmetry, no matter how largely it is broken, will manifest itself in a clear way at low energies, especially in the structure of Yukawa coupling matrices under the support of the non-renormalization theorem due to supersymmetry.

In this paper we make an attempt to understand generations based on the horizontal symmetry along the scenario of "spontaneous generation of generations". We work on the supersymmetric gauge theory where gauge group is the direct product of the horizontal symmetry G_H and the standard gauge group $SU(3) \times SU(2) \times U(1)$. The scenario requires us to introduce a noncompact group as a horizontal symmetry. Based on the simplest group $G_H = SU(1,1)$, we attempt a model building. We show how the hierarchical structure of Yukawa coupling matrices is realized in the scenario.

§2. Spontaneous generation of generations

Our basic hypothesis is that Nature is left-right symmetric at the fundamental level, and the gauge group $G_H \times SU(3) \times SU(2) \times U(1)$ governs the world in a vectorlike manner.[7] All particles, represented by left-handed chiral superfields, thus belong to the totally real representation of the gauge group. We expect that any left-right asymmetries realized at low energies come through the spontaneous breakdown of G_H.

Let Q be the chiral multiplet which belongs to some representation of G_H and has quantum numbers of quark doublet q under $SU(3) \times SU(2) \times U(1)$. Q contains as subcomponents three generations of q_i $(i = 1, 2, 3)$ as well as extra doublets q_{extra}. We then demand that there exists the chiral multiplet $\bar{Q}$ which belongs to the conjugate representation of Q under $G_H \times SU(3) \times SU(2) \times U(1)$. The scenario "spontaneous generation of generations" implies that all components $\bar{q}$s in $\bar{Q}$ and all q_{extra}s in Q acquire huge Dirac mass terms $\bar{q} \, q_{extra}$ through the spontaneous breakdown of G_H and decouple from low energy physics, retaining q_1, q_2 and q_3 massless. At a glance, this is impossible, because "vectorlike" implies that the total number of qs is equal to that

of $\bar{q}$s and that the appearance of massless qs is always accompanied by the appearance of the same number of massless $\bar{q}$s as far as the total number is finite. This is not a situation we seek for. The unique loophole which leads to the realization of the scenario is that Q and $\bar{Q}$ belong to the infinite dimensional representation of G_H.[8]

This is terrible, not only because we have infinite number of particles (like string theory) but also because the horizontal symmetry becomes noncompact if we stick to the unitary representation for the particles. It is unclear whether we can treat such type of noncompact gauge theory based on the conventional field theoretical framework or not. Nevertheless, we dare to proceed further, postponing a lot of things to future studies.

In order to extract the essential feature of the scenario as transparent as possible, we work in this paper on the simplest noncompact horizontal symmetry, $G_H = SU(1, 1)$.

The components of any representation of $SU(1.1)$ are labeled by weights, the eigenvalues of the third component H_3 of the $SU(1, 1)$ generators $\{H_1, \ H_2, \ H_3\}$ which form the algebra

$$[H_1, \ H_2] = -iH_3, \ [H_2, H_3] = iH_1, \ [H_3, H_1] = iH_2. \tag{4}$$

The unitary representations are infinite dimensional and classified to two types (positive and negative) depending on the sign of weights. The positive representation contains components whose weights run from some real positive number (lowest weight) α to infinity by one unit, $\{\alpha, \ \alpha + 1, \ \alpha + 2, \cdots\}$. The negative representation is the conjugate of the positive representation, and then the weights of components are $\{-\alpha, \ -\alpha - 1, \ -\alpha - 2, \cdots\}$ with the highest weight $-\alpha$.

Let us first assume that the quark doublets qs belong to the positive representation Q_α with the lowest weight α. Then $\bar{q}$s are assigned to the negative representation $\bar{Q}_{-\alpha}$ with same α:

$$\begin{aligned} Q_\alpha &= \{q_\alpha, \ q_{\alpha+1}, \ q_{\alpha+2}, \cdots\} \ , \\ \bar{Q}_{-\alpha} &= \{\bar{q}_{-\alpha}, \ \bar{q}_{-\alpha-1}, \ \bar{q}_{-\alpha-2}, \cdots\} \ . \end{aligned} \tag{5}$$

At this stage, the choice of sign of weights is a matter of convention. Once it is fixed by (5), however, the signs for other multiplets are physically relevant. As we will

see later, all quarks and leptons (q, $\bar{u}$, $\bar{d}$, ℓ, $\bar{e}$) appearing at low energies must be assigned to positive representations, and their conjugates ($\bar{q}$, u, d, $\bar{\ell}$, e) to negative representations. Thus we have following multiplets:

$$
\begin{aligned}
& Q_\alpha, \ \ \bar{U}_\beta, \ \ \bar{D}_\gamma, \ \ L_\eta, \ \ \bar{E}_\lambda \ , \\
& \bar{Q}_{-\alpha}, \ \ U_{-\beta}, \ \ D_{-\gamma}, \ \ \bar{L}_{-\eta}, \ \ E_{-\lambda} \ ,
\end{aligned}
\tag{6}
$$

where α, β, γ, η and λ are real positive numbers.

Now we introduce a multiplet Ψ which is responsible to the spontaneous breakdown of $SU(1,1)$, and couple it to quarks and leptons in the superpotential by

$$
x_Q \ Q_\alpha \ \bar{Q}_{-\alpha} \ \Psi + \ x_U \ \bar{U}_\beta \ U_{-\beta} \ \Psi + x_D \ \bar{D}_\gamma \ D_{-\gamma} \ \Psi + \ x_L \ L_\eta \ \bar{L}_{-\eta} \ \Psi + \ x_E \ \bar{E}_\lambda \ E_{-\lambda} \ \Psi \ . \tag{7}
$$

Ψ is assumed to be singlet under $SU(3) \times SU(2) \times U(1)$. As for $SU(1,1)$, the weights of its components are restricted to be integral. Suppose that the vacuum of the theory breaks $SU(1,1)$ through the non-vanishing vacuum expectation value ($v.e.v.$) $\langle \psi_{-g} \rangle \neq 0$ of some component ψ_{-g} of Ψ. Through the couplings (7), quarks and leptons get masses. For Q and $\bar{Q}$, for example, mass term is given by

$$
x_Q \ Q_\alpha \ \bar{Q}_{-\alpha} \langle \Psi \rangle = x_Q \ \langle \psi_{-g} \rangle \sum_{n=0}^{\infty} \ C_n^Q \ q_{\alpha+n+g} \ \bar{q}_{-\alpha-n}, \tag{8}
$$

where C_n^Q is the Clebsch-Gordan coefficient. It should be noticed that the $SU(1,1)$ invariance requires the additive conservation of the weights. Eq.(8) implies that the first g components of Q (q_α, $q_{\alpha+1}$, $\cdots$, $q_{\alpha+g-1}$) escape from acquiring mass and stay massless. All components of $\bar{Q}$ form Dirac mass terms with the remaining components of Q and become massive. Thus the choice $g = 3$ with $\langle \psi_{-3} \rangle \neq 0$ realizes just three generations of massless quark doublets. In this way, three generations of massless quarks and leptons (q, $\bar{u}$, $\bar{d}$, ℓ, $\bar{e}$) are realized as the staff members of low energy theory through coupling (7). We notice that the common sign choice for Q, $\bar{U}$, $\bar{D}$, L and $\bar{E}$ presented in (6) is essential for this result. If, for example, $\bar{U}$ were assigned to negative representation, u were realized, instead of $\bar{u}$, in the massless spectrum.

Up to now, we have not fixed the $SU(1,1)$ representation of Ψ. If we insist on the unitary nature of the representation, we may assign it to infinite dimensional representation. But this is not allowed. The $SU(1,1)$ invariance of the couplings (7) is realized only when Ψ is assigned to finite dimensional representation, which is consequently non-unitary. Thus we have

$$\Psi = \{\psi_{-R}, \ \psi_{-R+1}, \ \cdot \ \cdot \ \cdot \ , \ \psi_0, \ \cdot \ \cdot \ \cdot \ , \ \psi_{R-1}, \ \psi_R\} \ , \tag{9}$$

where the highest weight R must be three or more ($R = 3, \ 4, \ 5, \cdots$) so that Ψ contains the component ψ_{-3}. The remarkable feature we observe from the formula (A8) given in the Appendix is that the Clebsch-Gordan coefficient C_n^Q in Eq. (8) behaves as $C_n^Q \sim n^R$ for large n, and therefore the masses of qs and $\bar{q}$s blow up rapidly in n. We thus expect that the infinite number of redundant quarks and leptons will safely decouple from low energies.

The necessity of the introduction of the non-unitary representation for Ψ and originally of the introduction of the noncompact gauge group itself makes it decisive that we cannot work in the conventional framework of the renormalizable supersymmetric gauge theory. The canonical kinetic Lagrangian inevitably induces intractable negative norm states for those belonging to non-unitary representations. We must work on the theory with non-canonical and consequently non-renormalizable kinetic Lagrangian. We expect that the supergravity theory[9] gives well-defined working frame for such a theory. The metric K_i^j in the kinetic Lagrangian of Ψ, for example,

$$K_i^j \ \partial_\gamma \ \psi_i^* \ \partial^\gamma \ \psi_j \ , \tag{10}$$

is given in terms of the Kähler potential K by

$$K_i^j = \frac{\partial^2 K}{\partial \psi_i^* \partial \psi_j} \quad . \tag{11}$$

What is needed is that K_i^j is field dependent and its $v.\,e.\,v.\ \left\langle K_i^j \right\rangle$ is positive definite. In this context, it is reasonable to expect that $v.\,e.\,v.$ of ψ is of order of the Planck mass M_P.

The scenario may be phrased in the following way. Nature has the symmetry $SU(1,1)$ at the fundamental level. Leptons and quarks belong to infinite dimensional unitary representations. There also exist fields Ψ which belong to non-unitary representations. The vacuum inevitably breaks $SU(1,1)$ in order to realize well-defined positive norm states, and leads to non-vanishing $v.\,e.\,v.$ for Ψ. In the supersymmetric theory, it is a familiar experience that several supersymmetric vacua degenerate.[4] If the vacuum with $\langle \psi_0 \rangle \neq 0$ were realized, we would have no chiral generations. In this sense, chiral three generations are generated spontaneously through $\langle \psi_{-3} \rangle \neq 0$.

Here we emphasize the special role of supersymmetry in the scenario. If the theory does not have supersymmetry, Ψ is merely a scalar field belonging to real representation of $SU(1,1)$, and when the Yukawa coupling $Q\bar{Q}\Psi$ is allowed by gauge symmetry, the coupling $Q\bar{Q}\Psi^\dagger$ is also allowed. If both terms coexist, the chiral nature of the resulting mass terms is destroyed and all qs become massive. The supersymmetry forbids the latter coupling due to the chiral nature of $Q, \bar{Q}$ and Ψ.

§3. Higgs multiplets

Let us now proceed to the Higgs sector. Supersymmetry requires us to introduce two types of Higgs doublets h and h' in order to realize Yukawa couplings (2) in the superpotential.[4] Since all conventional quarks and leptons have positive weights of $SU(1,1)$, h and h' must belong to infinite dimensional representations with negative weights:

$$
\begin{aligned}
H_{-\rho} &= \{h_{-\rho},\ h_{-\rho-1},\ h_{-\rho-2}, \cdots \} \ , \\
H'_{-\sigma} &= \{ h'_{-\sigma}\ h'_{-\sigma-1},\ h'_{\sigma-2}, \cdots \} \ .
\end{aligned}
\tag{12}
$$

The $SU(1,1)$ invariance of the Yukawa couplings (in the superpotential)

$$
y_U\ \bar{U}_\beta\ Q_\alpha\ H_{-\rho} + y_D\ \bar{D}_\gamma\ Q_\alpha\ H'_{-\sigma} + y_E\ \bar{E}_\lambda\ L_\eta\ h'_{-\sigma}
\tag{13}
$$

restricts possible values of the weights as

$$\rho = \alpha + \beta + \Delta \ ,$$
$$\sigma = \alpha + \gamma + \Delta' = \eta + \lambda + \Delta'' \ , \tag{14}$$

where Δ, Δ' and Δ'' are non-negative integers $(0, 1, 2, \cdot \cdot \cdot)$. The Clebsch-Gordan decomposition of the first term of (13), for example, takes the form

$$y_U \ \bar{U}_\beta \ Q_\alpha \ H_{-\rho} = y_U \ \sum_{i,j=0}^{\infty} \ C_{i,j}^U \ \bar{u}_{\beta+i} \ q_{\alpha+j} \ h_{-\rho+\Delta-i-j} \ . \tag{15}$$

The left-right symmetry of the theory requires the existence of the conjugates

$$\bar{H}_\rho = \{\bar{h}_\rho, \ \bar{h}_{\rho+1}, \ \bar{h}_{\rho+2}, \cdots\}$$
$$\bar{H}'_\sigma = \{\bar{h}'_\sigma, \ \bar{h}'_{\sigma+1}, \ \bar{h}'_{\sigma+2}, \cdots\}, \tag{16}$$

which have Yukawa couplings

$$y_U^* \ U_{-\beta} \ \bar{Q}_{-\alpha} \ \bar{H}_\rho + y_D^* \ D_{-\gamma} \ \bar{Q}_{-\alpha} \ \bar{H}'_\sigma + y_E^* \ E_{-\lambda} \bar{L}_\eta \ \bar{H}'_\sigma \ . \tag{17}$$

According to the minimal supersymmetric standard model,[4] we wish to reproduce just one pair of Higgs doublets h and h' as the massless states. For this purpose we introduce a finite dimensional multiplet Φ with the highest weight S $(S = 1, 2, 3, \cdots)$,

$$\Phi = \{\phi_{-S}, \phi_{-S+1}, \ \cdots, \phi_0, \ \cdots, \phi_{S-1}, \phi_S\} \ , \tag{18}$$

and couple it to Higgs multiplets by

$$x_H \ H_{-\rho} \ \bar{H}_\rho \ \Phi + x_{H'} \ H'_{-\sigma} \ \bar{H}'_\sigma \ \Phi \ . \tag{19}$$

The nonvanishing $v.e.v.$ $\langle\phi_1\rangle \neq 0$ then picks up $h_{-\rho}$ and $h'_{-\sigma}$ as massless states, making all other components massive.

For the economical purpose, we might have coupled Ψ of (9) to $H\bar{H}$ and $H'\bar{H}'$ instead of Φ. In this case, however, the massless states are the first three components of $\bar{H}$ and $\bar{H}'$, which do not have couplings to massless quarks and leptons. So we need to introduce at least two multiplets Ψ and Φ as those which are responsible to the spontaneous breakdown of $SU(1,1)$.

In this way, all staffs of low energy supersymmetric standard model are prepared as the chiral massless states. They are

$$
\begin{array}{ll}
q_\alpha, \ q_{\alpha+1}, \ q_{\alpha+2} & \ell_\eta, \ \ell_{\eta+1}, \ell_{\eta+2} \\[4pt]
\bar{u}_\beta, \ \bar{u}_{\beta+1}, \ \bar{u}_{\beta+2} & \bar{e}_\lambda, \ \bar{e}_{\lambda+1}, \ \bar{e}_{\lambda+2} \\[4pt]
\bar{d}_\gamma, \ \bar{d}_{\gamma+1}, \ \bar{d}_{\gamma+2} & h_{-\rho}, \ h'_{-\sigma} \ .
\end{array}
\tag{20}
$$

The Clebsch-Gordan decomposition (15) and its counterparts for the other two couplings in (13) completely determine the Yukawa couplings of these massless states in terms of the Clebsch-Gordan coefficients $C_{i,j}^U$ etc. The detailed structure strongly depends on the values of Δ, Δ' and Δ'' in Eq.(14). For the u-quark coupling

$$
\sum_{i,j=0}^{2} \Gamma_u^{ij} \ \bar{u}_{\beta+i} \ q_{\alpha+j} \ h_{-\rho},
\tag{21}
$$

we obtain, up to overall normalization,

$$\Gamma_u(\Delta = 0) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_u(\Delta = 1) \sim \begin{pmatrix} 0 & -\sqrt{\beta} & 0 \\ \sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma_u(\Delta = 2) \sim \begin{pmatrix} 0 & 0 & \sqrt{\beta(2\beta+1)} \\ 0 & -\sqrt{(2\alpha+1)(2\beta+1)} & 0 \\ \sqrt{\alpha(2\alpha+1)} & 0 & 0 \end{pmatrix} \tag{22}$$

$$\Gamma_u(\Delta = 3) \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2\beta+1} \\ 0 & \sqrt{2\alpha+1} & 0 \end{pmatrix}$$

$$\Gamma_u(\Delta = 4) \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_u(\Delta \geq 5) \sim 0 \ .$$

We may be tempted to take the third case in (22) ($\Delta = \Delta' = \Delta'' = 2$) to realize the observed masses of quarks and leptons by tuning the values of weights ($\alpha, \beta, \cdots$). This, however, cannot give the non-trivial Cabibbo-Kobayashi-Maskawa matrix. Moreover, we primarily expect that the weights are order one quantity and are not extremely small nor extremely large. Thus none of the patterns in(22) looks realistic. In the next section, we will see that the possible existence of the $SU(1,1)$ invariant mass terms, which have been overlooked, gives significant modification. In this context, the patterns (22) are regarded as the zeroth order form of the Yukawa couplilng matrix of the model.

§4. $SU(1,1)$ invariant mass and generation mixing

We have constructed a model step by step and have arrived at the superpotential

$$W = x_{Q} \ Q \ \bar{Q} \ \Psi + x_{U} \ \bar{U} \ U\Psi + x_{D} \ \bar{D} \ D \ \Psi + x_{L} \ L \ \bar{L} \ \Psi + x_{E} \ \bar{E} \ E\Psi$$
$$+ x_{H} \ H \ \bar{H} \ \Phi + x_{H'} \ H' \ \bar{H}'\Phi$$
$$+ y_{U} \ \bar{U} \ Q \ H + y_{D} \ \bar{D} \ Q \ H' + y_{E} \ \bar{E} \ L \ H' \tag{23}$$
$$+ y_{U} \ U \ \bar{Q} \ \bar{H} + y_{D} \ D \ \bar{Q} \ \bar{H}' + y_{E} \ E \ \bar{L} \ \bar{H}'$$
$$+ \widetilde{W}(\Psi, \Phi) \ ,$$

where $\widetilde{W}$ represents possible self-couplings of Ψ and Φ. Without loss of generality, all coupling constants xs and ys are taken to be real under the left-right symmetry.

Let us now discuss the possible cubic couplings missing in the superpotential (23). The gauge invariance may allow other terms. The remarkable feature of the model is that the dangerous baryon-number and lepton-number non-conserving interactions enumerated in (3) are all forbidden by model construction. The couplings $LL\bar{E}, QL\bar{D}$ and $\bar{D}\bar{D}\bar{U}$ are absent simply because all multiplets have positive weights. The coupling $H'H'\bar{E}$ is forbidden by the second constraint of (14). Due to the vectorlike extension, the model generally admits new couplings

$$Q \ Q \ D + \bar{Q} \ \bar{Q} \ \bar{D} + Q \ \bar{U} \ \bar{L} + \bar{Q} \ U \ L \tag{24}$$

when the weights satisfy suitable relations. These terms, however, will be harmless, if exist, because they do not lead to the cubic couplings among massless states in (20). The truly dangerous cubic terms are

$$(L \ H \ + \ \bar{L} \ \bar{H}) \ (\Psi \ + \ \Phi)$$
$$+(Q \ \bar{Q} \ + \ \bar{U} \ U + \ \bar{D} \ D \ + \ L \ \bar{L} \ + \ \bar{E} \ E) \ \Phi \tag{25}$$
$$+(H \ \bar{H} + \ H' \ \bar{H}') \ \Psi \ .$$

These terms must be absent because they will seriously affect the massless spectrum of the model. Although the first term may be forbidden by imposing $|\eta - \rho| \neq integer$, the second and third terms can not be forbidden by the gauge principle. This is the most unpleasant aspect of the present $SU(1,1)$ model. We may also encounter the difficulty if we further admit the higher power couplings of Ψ and Φ to quarks, leptons and Higgses. For example the coupling $Q \ \bar{Q} \ \Psi^{n} \ (n \geq 2)$ may tend to reproduce $3n$ generations of qs,

and the mixture of different powers of Ψ makes the massless spectrum quite vague. At present, we have no good reasoning. We only expect that the absence of these terms is a manifestation of the profound feature of more fundamental theory which nontrivially controls the dynamics of Ψ and Φ.

In addition to the cubic couplings in (23), the $SU(1,1)$ invariance allows following mass terms:

$$
M_Q \ Q \ \bar{Q} + M_U \ \bar{U} \ U + M_D \ \bar{D} \ D + M_L \ L \ \bar{L} + M_E \ \bar{E} \ E
$$
$$
+ M_H \ H \ \bar{H} + M_{H'} \ H' \ \bar{H}' \ . \tag{26}
$$

We expect all these masses are of order M_P because we have no reason to suppress them. At a glance, one may suspect that these mass terms completely upset the scenario based on the couplings (7) and (19), because they give masses to all quarks, leptons and Higgses. As a matter of fact, the effect of (26) is to replace the original massless states (20) by the linear combinations which contain infinite number of higher-weight states. As we will see below, this mixing effect which utilizes the seesaw mechanism[10] gives a significant modification to the Yukawa coupling matrices[11] (22), which may solve the mystery of Yukawa coupling hierarchy.

Let us first discuss the Higgs sector $\{H_{-\rho}, \bar{H}_\rho\}$ Their mass terms consists of two parts; one is the $SU(1,1)$ invariant mass and the other is the mass due to the coupling to $\langle \Phi \rangle$,

$$
M_H \ H_{-\rho} \ \bar{H}_\rho + x_H \ H_{-\rho} \ \bar{H}_\rho \ \langle \Phi \rangle
$$
$$
= M_H \ \sum_{n=0}^{\infty} (-1)^n h_{-\rho-n} \ \bar{h}_{\rho+n} + x_H \ \langle \phi_1 \rangle \sum_{n=0}^{\infty} C_n^H \ h_{-\rho-n-1} \ \bar{h}_{\rho+n} \ . \tag{27}
$$

The important point we recall is that the Clebsch-Gordan coefficient C_n^H is a monotonically increasing function of n and blows up rapidly for large n by $C_n^H \sim n^S$. This fact assures, no matter how M_H is large, the existence of just one massless mode h which consists of a linear combination of $h_{-\rho-n}$ with rapidly decreasing coefficients,

$$
h = \sum_{n=0}^{\infty} a_n^{H*} \ h_{-\rho-n}. \tag{28}
$$

The coefficient a_n^H satisfies the recursion equation

$$M_H \ (-1)^n \ C_n^H + x_H \ \langle \phi_1 \rangle \ C_n^H \ a_{n+1}^H = 0 \qquad (29)$$

which assures the orthogonality of h to massive modes. This gives

$$a_n^H = \epsilon_H^n \ a_0^H \ \prod_{r=0}^{n-1} \frac{(-1)^r}{C_r^H} \quad , \qquad (30)$$

where

$$\epsilon_H = -\frac{M_H}{x_H \ \langle \phi_1 \rangle} \qquad (31)$$

and a_0^H is fixed by the normalization $\sum_{n=0}^{\infty} a_n^H \ a_n^{H*} = 1$.

The almost same discussion applies to the other Higgs sector $\{H'_{-\sigma}, \bar{H}'_{\sigma}\}$, and re-produces the massless mode h'. For quarks and leptons, the mass terms are

$$\begin{aligned}
&M_F \ F \ \bar{F} + \ x_F \ F \ \bar{F} \ \langle \Psi \rangle \\
=&M_F \ \sum_{n=0}^{\infty} (-1)^n \ f_{\alpha_F+n} \ \bar{f}_{-\alpha_F-n} + x_F \ \langle \psi_{-3} \rangle \ \sum_{n=0}^{\infty} C_n^F \ f_{\alpha_F+n+3} \ \bar{f}_{-\alpha_F-n} \quad ,
\end{aligned} \qquad (32)$$

where $F \equiv \{f_{\alpha_F}, f_{\alpha_F+1}, \cdots\}$ represents Q, $\bar{U}$, $\bar{D}$, L and $\bar{E}$. Eq.(32) reproduces three massless modes, each of which consists of a linear combination of the components with weights connected by three units,

$$f^{(i)} = \sum_{n=0}^{\infty} a_n^{F(i)*} \ f_{\alpha_F+3n+i} \quad , \qquad i = 0, \ 1, \ 2. \qquad (33)$$

The recursion equation similar to (29) gives

$$a_n^{F(i)} = \epsilon_F^n \, a_0^{F(i)} \prod_{r=0}^{n-1} \frac{(-1)^{3r+i}}{C_{3r+i}^F} \tag{34}$$

with

$$\epsilon_F \equiv -\frac{M_F}{x_F \, \langle \psi_{-3} \rangle} \quad . \tag{35}$$

Thus we have the modified version of the massless modes in one to one correspondence to the original ones given in (20). The Yukawa couplings of these modes are obtained by substituting the inverse relation of Eqs. (28) and (33) to Eq. (15) etc. For example, for $\bar{u} \, q \, h$ coupling

$$\sum_{i,j=0}^{2} \Gamma_u^{ij} \, \bar{u}^{(i)} \, q^{(j)} \, h \quad , \tag{36}$$

we have

$$\Gamma_u^{ij} = y_U \sum_{n,n'=0}^{\infty} C_{3n+i, \, 3n'+j}^U \, a_n^{\bar{U}(i)} \, a_{n'}^{Q(j)} \, a_{3(n+n')+i+j-\Delta}^{H} \quad , \tag{37}$$

where Δ is a non-negative integer determined by (14), and $a_{n<0}^H \equiv 0$ should be understood.

Eq.(37) gives the final formula for the Yukawa coupling matrix Γ_u. Other Yukawa coupling matrices Γ_d and Γ_e are obtained by an appropriate replacement of the weights appearing in $C_{i, \, j}$ and a_n.

§5. Yukawa coupling hierarchy

In §3, we obtained the zeroth order form of the Yukawa coupling matrix (22) which was the simplest manifestation of the $SU(1, \, 1)$ symmetry. In the last section, we showed how this matrix is modified under the existence of the $SU(1, \, 1)$ invariant mass terms (26), and derived the final formula (37). In this section, we argue how the formula is relevant to the observed mass hierarchy of quarks and leptons.

The characteristic feature of the formula (37) is, as can be seen from Eqs. (30) and (34), that the mixing coefficients a_ns behave as $a_n^F \sim \epsilon_F^n$ and $a_n^H \sim \epsilon_H^n$ with rapidly decreasing coefficients. If we take $\epsilon_F = \epsilon_H = 0$, Eq. (37) reduces to the zeroth order form (22). Therefore, for not so large value of ϵs ($\epsilon \lesssim 1$), Γ_u^{ij} can be safely expanded in terms of the power series of ϵs. Clearly the first nontrivial power of ϵ in each entry of Γ_u depends on the generation indices i and j ($= 0, 1, 2$). This means that the coupling matrix Γ_u^{ij} takes a hierarchical structure in generation space. In fact, retaining the lowest nonvanishing power term of ϵ in Eq. (37), we obtain

$$
\Gamma_u(\Delta = 0) \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon^2 & \epsilon^3 & \epsilon^4 \end{pmatrix}
$$

$$
\Gamma_u(\Delta = 1) \sim \begin{pmatrix} \epsilon^3 & 1 & \epsilon \\ 1 & \epsilon & \epsilon^2 \\ \epsilon & \epsilon^2 & \epsilon^3 \end{pmatrix}
$$

$$
\Gamma_u(\Delta = 2) \sim \begin{pmatrix} \epsilon^2 & \epsilon^3 & 1 \\ \epsilon^3 & 1 & \epsilon \\ 1 & \epsilon & \epsilon^2 \end{pmatrix} \tag{38}
$$

$$
\Gamma_u(\Delta = 3) \sim \begin{pmatrix} \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon^2 & \epsilon^3 & 1 \\ \epsilon^3 & 1 & \epsilon \end{pmatrix}
$$

$$
\Gamma_u(\Delta = 4) \sim \begin{pmatrix} \epsilon^4 & \epsilon & \epsilon^2 \\ \epsilon & \epsilon^2 & \epsilon^3 \\ \epsilon^2 & \epsilon^3 & 1 \end{pmatrix} \quad ,
$$

where we have set $\epsilon_H = \epsilon_Q = \epsilon_{\bar{U}} \equiv \epsilon$ and omitted the numerical coefficients.

The observed mass spectrum of quarks and leptons suggests that the first case in (38) is realized for all Yukawa couplings, that is, $\Delta = \Delta' = \Delta'' = 0$ in Eq.(14). Then the masses of the 1st, 2nd and 3rd generation quarks and leptons are $0(\epsilon^4)$, $0(\epsilon^2)$ and $0(1)$. The Cabibbo-Kobayashi-Maskawa matrix takes the form

$$
U_{CKM} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \quad . \tag{39}
$$

These results are certainly qualitatively reasonable when we recall the numerical factors

related to the Clebsch-Gordan coefficients.

The fact that the power ϵ^n appears in Eq.(37) accompanied by rapidly decreasing coefficient further strengthens the hierarchical structure of Γ_u due to ϵ^n itself. Let us discuss the case $\Delta = 0$ more precisely retaining numerical factors. In this case, the terms with $n > 0$ and/or $n' > 0$ in (37) always give the higher order corrections, and the leading contribution is given by the first term with $n = n' = 0$;

$$\Gamma_u^{ij} \sim y_U\, C_{i,j}^U\, a_{i+j}^H \quad . \tag{40}$$

The Clebsch-Gordan coefficients are read off from the formulas (A8) and (A9) given in the Appendix:

$$C_{i,\,j}^U = N\sqrt{\frac{(i+j)!\,\Gamma(2\alpha+j)\,\Gamma(2\beta+i)}{i!\,j!\,\Gamma(2\alpha+2\beta+i+j)}} \tag{41}$$

$$\begin{aligned}
C_n^H = N'(-1)^n &\sqrt{\frac{n!\,(n+1)!}{\Gamma(2\rho+n)\,\Gamma(2\rho+n+1)}} \\
&\times \sum_{r=0}^{S-1} \frac{\Gamma(2\rho+n+1+r)}{(S-r)!\,(S-r-1)!\,r!\,(r+1)!\,(r+1+n-S)!}
\end{aligned} \tag{42}$$

where $\rho = \alpha + \beta$, and N and N' are normalization constants. The integer S, which is the highest weight of the multiplet Φ, is restricted to positive integer. The growing feature of C_n^H with respect to n becomes much radical for larger value of S, but even in the smallest case $S = 1$, it is still sizable and realizes a remarkable hierarchy in the numerical coefficients in Γ_u^{ij} :

$$\Gamma_u^{ij} \sim y_U \epsilon_H^{i+j}\, \frac{\Gamma(2\alpha+2\beta)}{\Gamma(2\alpha+2\beta+i+j)}\sqrt{\frac{\Gamma(2\alpha+j)\,\Gamma(2\beta+i)}{i!\,j!\,\Gamma(2\alpha)\,\Gamma(2\beta)}} \quad , \tag{43}$$

where we have fixed the normalization constants N and N' such that $C_{0,\,0}^U = C_0^H = 1$. For example if we take $\alpha = \beta = 1/2$, we have

$$\Gamma_u^{ij} \sim y_U\, \frac{\epsilon_H^{i+j}}{(i+j+1)!} \quad . \tag{44}$$

The eigenvalues of this matrix are

$$
\begin{aligned}
&y_U \ (1 + 0(\epsilon_H^2)) \ , \\
&y_U \ (-\frac{1}{12} \ \epsilon_H^2 + 0(\epsilon_H^4)) \ , \\
&y_U \ (\frac{1}{720} \ \epsilon_H^4 + 0(\epsilon_H^6)) \ .
\end{aligned}
\tag{45}
$$

This shows that not so small value of ϵ_H, for example $\epsilon_H \sim 1/3$, is able to reproduce the observed large mass hierarchy of quarks and leptons.

§6. Discussions

In this paper we have made an attempt at the model building based on the noncompact horizontal gauge group $SU(1, 1)$ following the scenario of "spontaneous generation of generations". This scenario may give new insight on the origin of the chiral generations of quarks and leptons and their hierarchical Yukawa couplings. Although it is too premature to decide the viability of the scenario from the very limitted analysis presented here, we expect that the gross feature of the obtained results is the reflection of the general feature of the scenario. Further studies are much desired both from phenomenological and theoretical points of view.

The most important phenomenological problem is whether the Yukawa coupling matrix (37) really reproduces the observed masses of quarks and leptons and the Cabibbo-Kobayashi-Maskawa matrix U_{CKM} under the reasonable choice of the weights. This requires a comprehensive analysis retaining full freedom of the parameters of the model including the relative phase of $\langle \Psi \rangle$ and $\langle \Phi \rangle$ indispensable for the CP violating phase in U_{CKM}. To be precise, we must further discuss the mass term of Higgs multiplets h and h'. The minimal supersymmetric standard model requires the mass term $\mu h h'$ in the superpotential. Such a mass term is, however, protected to vanish by $SU(1, 1)$ even after it is spontaneously broken. The unique remedy will be to extend the model so that the extra light $SU(3) \times SU(2) \times SU(1)$ singlets s survive at low energies and couple to h and h' in the superpotential by shh'.

The theoretical situation is much less satisfactory. First of all, it is unclear whether we can treat, in a manner adopted here, the theory of supergravity with noncompact gauge symmetry containing infinite dimensional unitary multiplets as well as finite dimensional non-unitary ones. If such a theory exists, the consistency may require some

constraints on the detailed structure of the theory, especially on the possible values of the weights. Furthermore we simply assumed the $v.e.v.$s of Ψ and Φ following our will. In principle, they must be determined through the stationality condition of the bosonic potential, which demands the knowledge on the Kähler potential K as well as the super-potential W. The requirement that these $v.e.v.$s do not break the local supersymmetry and at the same time realize the positive definite metric for all particle states will impose severe constraint on the structure of K and W. The vacuum structure of the theory must be clarified.

In our analysis we implicitly assumed the minimal form of the Kähler potential[9] for quarks, leptons and Higgses. In general, Ψ and Φ may be allowed to couple to them freely in the form like $Q^\dagger f(\Psi, \Psi^\dagger, \Phi, \Phi^\dagger)Q$ as far as they do not disturb the positivity of the metric. In this case the Yukawa coupling matrix (37) receives the modification due to the wave function renormalization of Q, $\bar{U}$ and H. So we need to know the principle which determines the form of $f(\Psi, \Psi^\dagger, \Phi, \Phi^\dagger)$ in order to derive the fully reliable results.

The characterization of the model further requires the clarification of the gauge anomalies.[8] Since we are working on the vectorlike theory, we might not worry about the gauge anomaly. However, when $SU(1, 1)$ is spontaneously broken, theory becomes chiral through the coupling of Ψ and Φ to matters in the superpotential. The model has been constructed by hand so that the resulting chiral fields form the anomaly free sets. But who knew this? The consistency of the theory seems to restrict the structure of the superpotential beyond the gauge invariance at the classical level.

We expect that the future attempts push some of these problems and open the way to the deeper understanding of the origin of generations of quarks and leptons.

Acknowledgements

Appendix

In this appendix we give some formulas for the $SU(1,\ 1)$ invariants needed in the text.

The $SU(1,\ 1)$ invariant bi-linears are

$$Q_\alpha^\dagger Q_\alpha \equiv \sum_{n=0}^\infty q_{\alpha+n}^* \, q_{\alpha+n} \quad , \tag{$A1$}$$

$$\bar{Q}_{-\alpha}^\dagger \bar{Q}_{-\alpha} \equiv \sum_{n=0}^\infty \bar{q}_{-\alpha-n}^* \, \bar{q}_{-\alpha-n} \quad , \tag{$A2$}$$

$$Q_\alpha \bar{Q}_{-\alpha} \equiv \sum_{n=0}^\infty (-1)^n \, q_{\alpha+n} \, \bar{q}_{-\alpha-n} \quad , \tag{$A3$}$$

$$\Psi^\dagger \Psi \equiv \sum_{n=-R}^R (-1)^n \, \psi_n^* \, \psi_n \quad , \tag{$A4$}$$

$$\Psi \, \Psi' \equiv \sum_{n=-R}^R (-1)^n \, \psi_n \, \psi'_{-n} \quad , \tag{$A5$}$$

where Ψ and Ψ' are assumed to have the highest weight R.

For the cubic invariants

$$Q_\alpha \, \bar{Q}_{-\alpha} \, \Psi \equiv \sum_{i,j=0}^\infty A_{i,\ j} \, q_{\alpha+i} \, \bar{q}_{-\alpha-j} \, \Psi_{-i+j} \quad , \tag{$A6$}$$

$$\bar{U}_\beta \, Q_\alpha \, H_{-\rho} \equiv \sum_{i,j=0}^\infty B_{i,\ j} \, \bar{u}_{\beta+i} \, q_{\alpha+j} \, h_{-\rho+\Delta-i-j} \quad , \tag{$A7$}$$

where $\Delta \equiv \rho - \alpha - \beta$ is restricted to non-negative integers, the Clebsch-Gordan coefficients $A_{i,\ j}$ and $B_{i,\ j}$ are given by

$$A_{i,\ j} = N_A(-1)^j \sqrt{\frac{(i-j+R)! \ i! \ j! \ (-i+j+R)!}{\Gamma(2\alpha+i) \ \Gamma(2\alpha+j)}}$$

$$\times \sum_{r=0}^{i-j+R} \frac{\Gamma(2\alpha+j+r)}{(R-r)! \ (i-j+R-r)! \ r! \ (j-i+R)! \ (r+j-R)!} \tag{$A8$}$$

and

$$B_{i,\,j} = N_B(-1)^{i+j}\,\sqrt{\frac{i!\ j!\ \Gamma(2\beta+i)\ \Gamma(2\alpha+j)}{(i+j-\Delta)!\ \Gamma(2\rho+i+j-\Delta)}}$$
$$\times \sum_{r=0}^{\Delta}(-1)^r\,\frac{(i+j-\Delta)!}{(i-r)!\ (j+r-\Delta)!\ r!\ (\Delta-r)!\ \Gamma(2\beta+r)\ \Gamma(2\alpha-r+\Delta)} \tag{$A9$}$$

where N_A and N_B are normalization constants independent of i and j.

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